Midsemestral examination 2010
M.Math.IInd year
Number Theory — B.Sury

Q 1. Let a, b be relatively prime integers. Let p be a prime. Determine the GCD of $a-b$ and $\frac{a^p-b^p}{a-b}$.
Hint: For any prime q dividing $a-b$, write $a \equiv b \bmod q$.

OR

If $m\phi(m) = n\phi(n)$, prove that $m = n$.

Q 2. Determine all the units in the ring $\mathbf{Z}[\sqrt{-d}]$ where d is a square-free positive integer congruent to 1 mod 4. Further, show that this fails for each of the rings $\mathbf{Z}[\sqrt{d}]$ where d is a square-free positive integer congruent to -1 mod 4.

Q 3. Let p be an odd prime and a be a primitive root mod p such that $a^{p-1} \not\equiv 1 \bmod p^2$. Prove that a is a primitive root mod p^n for each n.

OR

Show that $\phi(x) = n!$ has a solution in x for any natural number n.
Hint: Check that $n!$ is a value of r such that the equation $\phi(x) = r$ has a solution with x and r having the same prime factors.

Q 4. Use the quadratic reciprocity law to deduce that if p is any odd prime and $q \equiv 3 \bmod 4$ is a prime, then q is a quadratic residue mod p if and only if $p \equiv \pm a^2 \bmod 4q$ for some odd a relatively prime to q.

OR

Find all primes p such that 10 is a quadratic residue.

Q 5. Let $p = a^2 + b^2$ be an odd prime with $a \equiv 1 \bmod 4$. Prove that $a + b$ is a quadratic residue mod p if and only if $(a+b)^2 \equiv 1 \bmod 16$.

OR

Let a be an integer which is not a perfect square. Prove that there are infinitely many primes p such that n is not a square modulo p.

Q 6. Using the Chinese remainder theorem or otherwise, prove that there are arbitrarily long strings of consecutive natural numbers on which the Möbius function takes the same value.

OR

Prove that $\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\phi(d)}$ for all n.